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TECHNICAL REPORT

SYNTHESIS OF INTEGRAL MINIMAL WEIGHTS

by

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WORK CARRIED OUT AS PART OF THE LOCKHEED INDEPENDENT RESEARCH PROGRAM

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FOREWORD

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ABSTRACT

This report describes an effective and mathematically rigorous finite process for determining whether or not a given regular switching function is linearly separable, and if it is, for deriving a minimal set of integral weights and threshold to realize the function. The procedure uses the integral linear programming recently developed by R. E. Gomery; the algorithm is described in detail, and a corresponding computer program for implementing the technique can be obviously given without difficulty.

As an illustrative example, a regular switching function of nine variables is worked in detail. (The same function was studied earlier by D. G. Willis in disproving a conjecture.)

CONTENTS

Section 1
INTRODUCTION

The synthesis and minimization problem for linearly separable switching functions is a fundamental problem in threshold logic. The problem is to determine whether or not a given switching function is linearly separable and, if it is, to find the most economic system of weights and threshold. In our previous reports (Refs. 1-7), various methods have been applied to solve this problem.

For most of the threshold devices realizing linearly separable switching functions, the weights and the threshold are required to be integers. However, the optimal systems of weights and threshold obtained by the methods developed in the previous reports may fail to be integral; in fact, D. G. Willis (Ref. 8) discovered a linearly separable switching function of 9 variables of which

$$(2,\ 2,\ 3,\ 5,\ 5,\ 6.5,\ 6.5,\ 6.5,\ 9;\ 12)$$

is the only minimal system of weights and threshold. This disproves the conjecture of Elgot and Muroga (Ref. 9), and Muroga, Toda and Takasu (Ref. 10), about the existence of minimal weights which are all integers.

Because of this, it is desirable to find integral systems of weights and threshold that are the most economic among a given linearly separable function F. This problem will be referred to as the integral minimization problem. R. O. Winder (Ref. 11) proposed an approach and also mentioned several drawbacks. The problem remained unsolved as Winder remarked in Ref. 11.

The objective of the present report is to develop an application of the integral linear programming techniques introduced by R. E. Gomery (Ref. 12 and 13) to the integral minimization problem for linearly separable switching functions.

In the sections 2–4, we give an elementary exposition of Gomery's theory together with a modified finiteness proof so that it holds for our integral minimization problem. In section 5, we give an illustrative example which is used in the later part of the report. In the sections 6–9, we formulate the synthesis and minimization problem and then apply the method developed in sections 2–4 to solve the integral minimization problem. The Willis example previously mentioned is used as an illustration, and a minimal system of integral weights and threshold is computed by this process. The result is

$$(2,\ 2,\ 3,\ 5,\ 6,\ 7,\ 7,\ 9;\ 12)$$

Section 2
STANDARD LINEAR PROGRAMS

A standard linear program is to find nonnegative real numbers that minimize (or maximize) a given linear function subject to a given system of linear inequalities. Since a maximum problem is reduced to a minimum problem simply by multiplying the given linear function with -1, we may study the standard minimum program only.

For this purpose, let us consider a given linear function

$$a_{o,o} + \sum_{j=1}^{q} a_{o,j} t_j \tag{1}$$

of q variables $t_1, \ldots, t_q$, where $a_{o,o}$, $a_{o,1}$, $\ldots$, $a_{o,q}$ are given real numbers. On the other hand, let

$$a_{i,o} + \sum_{j=1}^{q} a_{i,j} t_j \geqq 0 \tag{2}$$

where $i = 1, 2, \ldots, p$, be a given system of p inequalities in the same variables t_1, $\ldots$, t_q with given real coefficients $a_{i,j}$ and constant terms $a_{i,o}$. Then, the standard linear minimum program is the problem of finding nonnegative real numbers $t_1, \ldots,$ t_q that minimize the given linear function (1) subject to the system (2) of p linear inequalities.

Now, let y denote the given linear function (1) and let x_i, $(i = 1, 2, \ldots, p)$, denote the linear function on the left side of the inequality (2). Consider $y, x_1, \ldots, x_p$ also as variables. Then, the standard linear minimum program described above can be restated as follows.

Find the minimum value of the variable y subject to the following $p + 1$ linear equations

$$y = a_{o,o} + \sum_{j=1}^{q} a_{o,j} t_j \tag{3}$$

$$x_i = a_{i,o} + \sum_{j=1}^{q} a_{i,j} t_j \tag{4}$$

where $i = 1, 2, \ldots, p$, and the condition that

$$x_i \geqq 0, \ (i = 1, 2, \ldots, p) \tag{5}$$

$$t_j \geqq 0, \ (j = 1, 2, \ldots, q) \tag{6}$$

Let A denote the $p + 1$ by $q + 1$ matrix

$$A = \| a_{i,j} \| \tag{7}$$

where $0 \leqq i \leqq p$ and $0 \leqq j \leqq q$. The matrix A is said to be dually feasible in case the q columns

$$\alpha_j = (a_{o,j}, a_{i,j}, \ldots, a_{p,j})$$

where $j = 1, 2, \ldots, q$, are lexicographically positive as defined in Ref. 7. In particular, A is dually feasible if

$$a_{o,j} > 0, \qquad (j = 1, 2, \ldots, q)$$

If the matrix A is dually feasible, then our standard linear minimum program is also said to be dually feasible.

Now, let us assume that our standard linear minimum program is dually feasible. Then, the dual simplex method of C. E. Lemke (Ref. 14) as formulated in Ref. 7 can be applied to obtain an optimal solution of the program or to prove the nonexistence of such.

If our standard linear minimum program has an optimal solution, then, after a finite number of pivoting operations as described in Ref. 7, the dual-simplex method reduces our given linear minimum program to the problem of finding the minimum value of the variable y subject to the following $p + q + 1$ linear equations

$$y = b_{o,o} + \sum_{j=1}^{q} b_{o,j} u_j \tag{8}$$

$$x_i = b_{i,o} + \sum_{j=1}^{q} b_{i,j} u_j \tag{9}$$

where $i = 1, 2, \ldots, p + q$, and the condition that

$$x_i \geqq 0, \quad (i = 1, 2, \ldots, p + q) \tag{10}$$

Here, the symbols x_i, $(p < i \leqq q)$, are used to denote the given nonbasic variables $t_1, \ldots, t_q$ with

$$t_j = x_{p+j}, \quad (j = 1, 2, \ldots, q)$$

and the new nonbasic variables $u_1, \ldots, u_q$ also appear on the left members of (9) as q of the $p + q$ variables $x_1, \ldots, x_{p+q}$. Furthermore, the $p + q + 1$ by $q + 1$ matrix

$$B = \| b_{i,j} \|$$

$(0 \leqq i \leqq p+q,\ 0 \leqq j \leqq q)$, remains dually feasible and satisfies the condition

$$b_{i,o} \geqq 0, \quad (1 \leqq i \leqq p+q) \tag{11}$$

Because of the condition (11), the trial solution

$$y = b_{o,o} \tag{12}$$

$$x_i = b_{i,o}, \quad (i = 1, 2, \ldots, p+q) \tag{13}$$

is an optimal solution of our standard linear minimum program.

Let α_o and β_o denote the $(p+q+1)$ vectors

$$\alpha_o = (a_{o,o}, a_{1,o}, \ldots, a_{p,o}, 0, \ldots, 0)$$

$$\beta_o = (b_{o,o}, b_{1,o}, \ldots, b_{p,o}, b_{p+1,o}, \ldots, b_{p+q,o})$$

Then, by (33) of Ref. 7, β_o is greater than α_o in the lexicographical order unless the given matrix A satisfies the condition

$$a_{i,o} \geqq 0 \quad (1 \leqq i \leqq p) \tag{14}$$

In case (14) is satisfied, then we may take $\beta_o = \alpha_o$.

Section 3
INTEGRAL LINEAR PROGRAMMING

In many practical linear programs, the optimal solutions are required to be integers. For example, in the classical transportation problem, if the commodity to be shipped is some indivisible good such as a number of automobiles, then the constraints and function to be minimized are just as usual but only integral values of the variables are admissible.

In 1958, R. E. Gomery (Refs. 12 and 13) introduced a general method for integral solutions to linear programs. A simple exposition of Gomery's algorithm with some slight improvement in the finiteness proof is given in this section. A numerical illustrative example is worked out in the following section.

For this purpose, let us assume that the coefficients and constant terms

$$a_{i,j}, \quad (0 \leqq i \leqq p, \ 0 \leqq j \leqq q)$$

in (1) and (2) are integers. Then, the <u>standard integral linear minimum program</u> is the problem of finding nonnegative integers $t_1, \ldots, t_q$ which minimize the linear function (1) subject to the system (2) of linear inequalities.

Introducing new variables $y, x_1, \ldots, x_p$ as in the preceding section, we obtain an equivalent problem of finding integers $t_1, \ldots, t_q$ which minimize the variable y subject to the $p+1$ linear equations (3) and (4) and the condition (5) and (6). Since the numbers $a_{i,j}$, $(0 \leqq i \leqq p, \ 0 \leqq j \leqq q)$, are assumed to be integers, and since the variables $t_1, \ldots, t_q$ are required to be integers, it follows that the variables $y, x_1, \ldots, x_p$ must also have integral values given by the equations (3) and (4). Hence, our standard integral linear program reduces to the problem stated as follows:

<u>Find nonnegative integers</u> $y, x_1, \ldots, x_p, t_1, \ldots, t_q$ <u>which minimize</u> y <u>and satisfy the</u> $p+1$ <u>linear equations</u> (3) <u>and</u> (4) <u>in section 2</u>.

Now, let us consider our problem as a usual standard linear program and assume that it is dually feasible as defined in section 2. Hence, we can apply the dual-simplex method to our program. The result is that either the equations (3) and (4) have no nonnegative solution whatsoever or our program has an optimal solution that is not necessarily integral.

If the equations (3) and (4) have no nonnegative solution, then our given standard integral linear minimum program certainly has no optimal solution in integers. Therefore, our problem is solved negatively in this case.

On the other hand, if our program has an optimal solution that is not necessarily integral, then the dual-simplex method reduces the given program to the problem of finding nonnegative integers

$$y, x_1, \ldots, x_{p+q}$$

which minimize y and satisfy the equations (8) and (9) as described in section 2.

If all of the $p + q + 1$ real numbers in the leading column

$$\beta_o = (b_{o,o}, b_{1,o}, \ldots, b_{p+q,o})$$

of the matrix B of the equations (8) and (9) are integers, then the trial solution (12) and (13) is an optimal integral solution of our given program. Therefore, our problem is solved affirmatively in this case.

In the remainder of this section, we shall consider the case in which not all of the $p + q + 1$ numbers in the leading column β_o of the matrix B are integers. Then,

the trial solution (12) and (13) is an optimal solution of our given program, but unfortunately it is not integral.

Choose the first entry $b_{i,o}$ in β_o from top down which fails to be an integer; in other words, let i_o denote the smallest nonnegative integer such that $b_{i_o,o}$ is not an integer. Consider the equation

$$x_{i_o} = b_{i_o,o} + \sum_{j=1}^{q} b_{i_o,j} u_j \tag{15}$$

where x_{i_o} stands for y in case $i_o = 0$.

Let h_o denote the smallest integer not less than $b_{i_o,o}$, and, for each $j = 1, 2, \ldots, q$, let h_j denote the largest integer not greater than $b_{i_o,j}$. Let

$$s_j = \begin{cases} h_o - b_{i_o,o}, & (\text{if } j = 0) \\ b_{i_o,j} - h_j, & (\text{if } j = 1, 2, \ldots, q) \end{cases}$$

Then we have

$$b_{i_s,j} = \begin{cases} h_o - s_o, & (\text{if } j = 0) \\ h_j + s_j, & (\text{if } j = 1, 2, \ldots, q) \end{cases}$$

with $0 \leqq s_j < 1$ for each $j = 0, 1, 2, \ldots, q$.

Substituting these values of $b_{i_o,j}$ into (15), we obtain

$$x_{i_o} - h_o - \sum_{j=1}^{q} h_j u_j = - s_o + \sum_{j=1}^{q} s_j u_j \tag{16}$$

For any integral solution of our program, $u_1, \ldots, u_q$ and x_{i_o} are integers. On the other hand, $h_o, h_1, \ldots, h_q$ are integers by definition. Hence, the left member

$$x_{p+q+1} = x_{i_o} - h_o - \sum_{j=1}^{q} h_j u_j \tag{17}$$

of the equation (16) must be an integer. Furthermore, since

$$s_o < 1, \qquad \sum_{j=1}^{q} s_j u_j \geqq 0$$

the right member of (16) must be greater than -1. Thus, the integer x_{p+q+1} is nonnegative and we obtain an equation

$$x_{p+q+1} = -s_o + \sum_{j=1}^{q} s_j u_j \tag{18}$$

Now, let us adjoin this additional equation (18) to the system (9) and let

$$b_{p+q+1,\, j} = \begin{cases} -s_o, & (\text{if } j = 0) \\ s_j, & (\text{if } j = 1, 2, \ldots, q) \end{cases}$$

Thus, we obtain the following system of $p + q + 1$ linear equations

$$x_i = b_{i,o} + \sum_{j=1}^{q} b_{i,j} u_j \tag{19}$$

where

$$i = 1, 2, \ldots, p + q + 1.$$

By the construction of (18), we have proved that, for any nonnegative integral solution $(x_1, \ldots, x_{p+q})$ of the system (9), there exists a nonnegative integer x_{p+q+1} such that $(x_1, \ldots, x_{p+q}, x_{p+q+1})$ is a nonnegative integral solution of (19). Conversely, if $(x_1, \ldots, x_{p+q}, x_{p+q+1})$ is any nonnegative integral solution of (19), then $(x_1, \ldots, x_{p+q})$ is clearly a nonnegative integral solution of the system (9).

Therefore, our standard integral linear program reduces to the problem of finding nonnegative integers

$$y, x_1, \ldots, x_{p+q}, x_{p+q+1}$$

which minimize y and satisfy the $p + q + 2$ linear equations (8) and (19).

Since the system (8) and (9) is dually feasible and since $s_j \geqq 0$ for each $j = 1, 2, \ldots, q$, it follows that the system (8) and (19) is also dually feasible. On the other hand, since $b_{p+q+1,o} = -s_o$ is negative, the trial solution of (8) and (19) is not nonnegative. Thus, we can apply the dual-simplex method to this new system (8) and (19). In fact, the first new nonbasic variable to be brought in is the variable x_{p+q+1}.

The result of the dual-simplex method will either prove that (19) has no nonnegative solution, or give an optimal solution of (8) and (19) that may fail to be integral.

If the system (19) has no nonnegative solution, then it certainly has no nonnegative integral solution. It follows that the system (9) also has no nonnegative integer solution. This solves our standard integral linear program negatively in this case.

On the other hand, if (8) and (19) have an optimal solution that is not necessarily integral, then, after a finite number of pivoting operations, the dual-simplex method

reduces (8) and (19) to the system of $p + q + 2$ linear equations

$$y = c_{o,o} + \sum_{j=1}^{q} c_{o,j} v_j \tag{20}$$

$$x_i = c_{i,o} + \sum_{j=1}^{q} c_{i,j} v_j \tag{21}$$

with $i = 1, 2, \ldots, p + q + 1$ and satisfying the condition

$$c_{i,o} \geqq 0, \quad (1 \leqq i \leqq p + q + 1) \tag{22}$$

Here, the new nonbasic variables $v_1, \ldots, v_q$ also appear on the left members of (21) as q of the $p + q + 1$ variables $x_1, \ldots, x_{p+q+1}$.

Let β_o and γ_o denote the $(p+q+1)$ vectors

$$\beta_o = (b_{o,o}, b_{1,o}, \ldots, b_{p+q,o})$$

$$\gamma_o = (c_{o,o}, c_{1,o}, \ldots, c_{p+q,o})$$

Then it follows from the dual feasibility and the dual-simplex method that $\gamma_o \geqq \beta_o$ in the lexicographical order.

Because of (22), the trial solution

$$y = c_{o,o} \tag{23}$$

$$x_i = c_{i,o} \tag{24}$$

with $i = 1, 2, \ldots, p + q + 1$, is an optimal solution of (8) and (19). Therefore, it follows that

$$y = c_{o,o} \tag{25}$$

$$x_i = c_{i,o}, \quad (i = 1, 2, \ldots, p + q) \tag{26}$$

is a feasible solution (8) and (9).

If $c_{o,o}, c_{1,o}, \ldots, c_{p+q,o}$ are integers, then $c_{p+q+1,o}$ is also an integer according to the construction of the last equation in the system (19). In this case, the trial solution (23) and (24) is an optimal integral solution of (8) and (19); hence (25) and (26) is an optimal integral solution of (8) and (9). Thus, our standard integral linear program is solved affirmatively.

It remains to study the case where not all of the numbers $c_{o,o}, c_{1,o}, \ldots, c_{p+q,o}$ are integers.

Choose the first entry $c_{i,o}$ in γ_o from top down which fails to be an integer; that is, let i_o denote the smallest nonnegative integer such that $c_{i_o,o}$ is not an integer. Consider the equation

$$x_{i_o} = c_{i_o,o} + \sum_{j=1}^{q} c_{i_o,j} v_j \tag{27}$$

where x_{i_o} stands for y in case $i_o = 0$.

Now, we can iterate our operation on (27) and the system (20) and (21) exactly as we did on (15) and the system (8) and (9).

In the next section, we prove that the process stops after a finite number of iterations provided that either of two mild conditions holds. The final result is that either the given program has no integral feasible solution or this process gives an optimal integral solution in a finite number of iterations.

Section 4
FINITENESS PROOF

Throughout the present section, we shall assume that at least one of the following two conditions (A) and (B) is satisfied:

(A) The nonnegative solutions of the system (2) of p inequalities form a bounded set S of the q-dimensional Euclidean space R^q.

(B) The system (2) has a nonnegative solution in integers, and the coefficients, $a_{o,1}, \ldots, a_{o,q}$ in (1) are all positive, that is,

$$a_{o,j} > 0, \qquad (j = 1, \ldots, q)$$

Under this additional assumption, we can prove that the process which is described in the preceding section must stop after a finite number of iterations.

For this purpose, let us assume the contrary and deduce the following contradiction. (Then the iterative process in the preceding section would apply infinitely many times.) Let $k \geqq 0$ be any nonnegative integer. After the k^{th} iteration of the process, our given standard integral linear program reduces to the problem of finding nonnegative integers

$$y, x_1, \ldots, x_{p+q+k}$$

which minimize y and satisfy the system

$$y = d_{o,o}^{(k)} + \sum_{j=1}^{q} d_{o,j}^{(k)} w_j^{(k)} \tag{28}$$

$$x_i = d_{i,o}^{(k)} + \sum_{j=1}^{q} d_{i,j}^{(k)} w_j^{(k)} \tag{29}$$

with $i = 1, 2, \ldots, p+q+k$ and satisfying the condition

$$d_{i,o}^{(k)} \geqq 0, \quad (1 \leqq i \leqq p+q+k) \tag{30}$$

Here, the nonbasic variables $w_1^{(k)}, \ldots, w_q^{(k)}$ also appear on the left members of (29) as q of the $p+q+k$ variables $x_1, \ldots, x_{p+q+k}$. Note that (28) and (29) reduce to (8) and (9) when $k = 0$ and to (20) and (21) when $k = 1$.

Let $\delta_o^{(k)}$ denote the $(p+q+1)$ vector

$$\delta_o^{(k)} = \left(d_{o,o}^{(k)}, d_{1,o}^{(k)}, \ldots, d_{p+q,o}^{(k)}\right)$$

Then it follows from the dual feasibility and the dual-simplex method that

$$\delta_o^{(k-1)} \leqq \delta_o^{(k)} \tag{31}$$

in the lexicographical order for every $k \geqq 1$.

In particular, (31) implies

$$d_{o,o}^{(k-1)} \leqq d_{o,o}^{(k)} \tag{32}$$

for every $k \geqq 1$. In other words, the infinite sequence

$$d_{o,o}^{(o)}, d_{o,o}^{(1)}, \ldots, d_{o,o}^{(k)}, \ldots \tag{33}$$

is monotone nondecreasing.

Next, we can prove that, under our additional assumption made at the beginning of this section, the infinite sequence (33) of real numbers is bounded above.

In fact, if the condition (A) is satisfied, then the continuous function (1) is bounded for all feasible solutions of (2). Since $d^{(k)}_{o,o}$ is the value of (1) for the feasible solution

$$t_j = x_{p+j} = d^{(k)}_{p+j,o}$$

where $j = 1, 2, \ldots, q$, the sequence (33) is bounded.

On the other hand, if the condition (B) is satisfied, then the system (2) has a non-negative integral solution

$$t_j = \tau_j, \quad (j = 1, 2, \ldots, q)$$

and the linear function (1) has the number

$$\eta = a_{o,o} + \sum_{j=1}^{q} a_{o,j}\,\tau_j$$

as its value for this solution of (2). Since the coefficients $a_{i,j}$ are assumed to be integral, it follows that

$$y = \eta = a_{o,o} + \sum_{j=1}^{q} a_{o,j}\,\tau_j$$

$$x_i = \xi = a_{i,o} + \sum_{j=1}^{q} a_{i,j}\,\tau_j$$

where $i = 1, 2, \ldots, p$ is an integral feasible solution of (3) and (4). Let

$$\xi_{p+j} = \tau_j \quad (j = 1, 2, \ldots, q)$$

then $y = \eta$, $x_i = \xi_i$ $(i = 1, 2, \ldots, p+q)$ is an integral feasible solution of (8) and (9).

By the construction of the iterative process in the preceding section, there exists a nonnegative integer ξ_{p+q+k} for each $k \geqq 1$ such that

$$y = \eta, \qquad x_i = \xi_i \qquad (i = 1, 2, \ldots, p+q+k)$$

is a feasible solution of (28) and (29). Since

$$y = d_{o,o}^{(k)}, \qquad x_i = d_{i,o}^{(k)} \qquad (i = 1, 2, \ldots, p+q+k)$$

is an optimal solution of (28) and (29), we have

$$d_{o,o}^{(k)} \leqq \eta$$

Hence the sequence (33) is bounded above in this case.

Thus, we have proved that the sequence (33) is bounded above provided that at least one of the two conditions (A) and (B) holds.

As a nondecreasing sequence which is bounded above, the sequence (33) converges to its least upper bound e_o, in symbols

$$\lim_{k \to \infty} d_{o,o}^{(k)} = e_o \tag{34}$$

Let h_o denote the smallest integer not less than e_o, then we have

$$e_o \leqq h_o \qquad h_o - e_o < 1$$

By (34), there exists a nonnegative integer k_o such that

$$s_o = h_o - d_{o,o}^{k_o} < 1$$

On the other hand, since e_o is the least upper bound of the sequence (33), we have

$$s_o = h_o - d_{o,o}^{(k_o)} \geqq 0$$

If $s_o = 0$, then we have

$$d_{o,o}^{(k_o)} = e_o = h_o$$

Hence $e_o = h_o$ is an integer and

$$d_{o,o}^{(k)} = e_o = h_o, \qquad (k > k_o) \tag{35}$$

Otherwise, if $s_o > 0$, then

$$d_{o,o}^{(k_o)} = h_o - s_o$$

is not an integer. According to the rule of choosing an equation in the iterative process, the equation (28) is to be chosen to construct the new equation

$$x_{p+q+k_o+1} = -s_o + \sum_{j=1}^{q} s_j\, w_j^{(k_o)} \tag{36}$$

where

$$s_j = d_{o,j}^{(k_o)} - h_j, \qquad (j = 1, 2, \ldots, q)$$

with h_j standing for the largest integer not greater than $d_{o,j}^{(k_o)}$.

Adjoin the equation (36) to the system (28) and (29) with $k = k_o$ and apply the dual-simplex method to the system (28), (29), and (36). By assumption, we will obtain an optimal solution given by the vector $\delta_o^{(k_o+1)}$. Since $-s_o$ is the only negative constant term in the system (28), (29), and (36), the first pivoting in the dual-simplex

method is to replace one of the nonbasic variables $w_1^{(k_o)}, \ldots, w_q^{(k_o)}$ by the new variable x_{p+q+k_o+1}. If, according to the rule of the dual-simplex method, the variable $w_{j_o}^{(k_o)}$ is selected to be replaced by x_{p+q+k_o+1}, then equations (28), (29), and (36) will be transformed into the form:

$$y = f_{o,o} + \sum_{j=1}^{q} f_{o,j} z_j \tag{37}$$

$$x_i = f_{i,o} + \sum_{j=1}^{q} f_{i,j} z_j \tag{38}$$

with $i = 1, 2, \ldots, p+q+k_o+1$, where the new nonbasic variables $z_1, \ldots, z_q$ are given by

$$z_j = \begin{cases} w_j^{(k_o)}, & (\text{if } j \neq j_o) \\ x_{p+q+k_o+1}, & (\text{if } j = j_o) \end{cases}$$

According to the pivoting process of the dual-simplex method, we have

$$f_{o,o} = d_{o,o}^{(k_o)} + \frac{s_o}{s_{j_o}} d_{o,j_o}^{(k_o)} \tag{39}$$

Because of the dual feasibility of the system (28) and (29), $d_{o,j_o}^{(k_o)} \geq 0$ and hence $s_{j_o} \leq d_{o,j_o}^{(k_o)}$. Therefore, (39) implies that

$$f_{o,o} \geq d_{o,o}^{(k_o)} + s_o = h_o .$$

On the other hand, since

$$f_{o,o} \leqq d_{o,o}^{(k)} \leqq e_o \leqq h_o, \quad (k > k_o)$$

it follows that

$$f_{o,o} = d_{o,o}^{(k)} = e_o = h_o, \quad (k > k_o)$$

Thus, we have proved that the limit e_o is an integer and that there exists an integer k_o such that

$$d_{o,o}^{(k)} = e_o, \qquad (k > k_o)$$

Now, let us consider those vectors $\delta_o^{(k)}$ with $k > k_o$. Since

$$\delta_o^{(k)} \leqq \delta_o^{(k+1)}, \quad d_{o,o}^{(k)} = d_{o,o}^{(k+1)}$$

for every $k > k_o$, we have

$$d_{1,0}^{(k)} \leqq d_{1,0}^{(k+1)} \tag{40}$$

for every $k > k_o$. In other words, the infinite sequence

$$d_{1,0}^{(k_o+1)}, d_{1,0}^{(k_o+2)}, \ldots, d_{1,0}^{(k)}, \ldots \tag{41}$$

is monotone nondecreasing.

We will prove that, under our additional assumption made at the beginning of this section, the sequence (41) is bounded.

If the condition (A) is satisfied, then the continuous function

$$x_1 = a_{1,0} + \sum_{j=1}^{q} a_{1,j} t_j \tag{42}$$

is bounded for all feasible solutions of (2). Since $d_{1,0}^{(k)}$ is the value of (42) for the feasible solution

$$t_j = x_{p+j} = d_{p+j,o}^{(k)}$$

where $j = 1, 2, \ldots, q$, the sequence (41) is bounded.

On the other hand, if the condition (B) is satisfied, then we have

$$a_{o,j} > 0, \qquad (j = 1, 2, \ldots, q)$$

For each $k > k_o$, we have

$$e_o = d_{o,o}^{(k)} = a_{o,o} + \sum_{j=1}^{q} a_{o,j}\, d_{p+j,o}^{(k)}$$

This implies that

$$d_{p+j,o}^{(k)} \leqq (e_o - a_{o,o})/a_{o,j}$$

for every $j = 1, 2, \ldots, q$ and therefore

$$0 \leqq d_{1,0}^{(k)} \leqq |a_{1,0}| + \sum_{j=1}^{q} |a_{1,j}| (e_o - a_{o,o})/a_{o,j}$$

for every $k > k_o$. Hence the sequence (41) is bounded.

Thus, we have proved that the sequence (41) is bounded when at least one of the two conditions (A) and (B) holds.

As a nondecreasing bounded sequence, (41) converges to its least upper bound e_1, in symbols

$$\lim_{k \to \infty} d_{1,0}^{(k)} = e_1 \tag{43}$$

Let h_o' denote the smallest integer not less than e_1, then we have

$$e_1 \leqq h_o', \quad h_o' - e_1 < 1$$

By (43), there exists a nonnegative integer $k_1 > k_o$ such that

$$s_o' = h_o' - d_{1,0}^{(k_1)} < 1$$

On the other hand, since e_1 is the least upper bound of the sequence (41), we have

$$s_o' = h_o' - d_{1,0}^{(k_1)} \geqq 0$$

If $s_1 = 0$, then we have

$$d_{1,0}^{(k_1)} = e_1 = h_0'$$

Hence $e_1 = h'_o$ is an integer and

$$d^{(k)}_{1,0} = e_1 = h'_o, \qquad (k > k_1) \tag{44}$$

Otherwise, if $s'_o > 0$, then

$$d^{(k_1)}_{1,0} = h'_o - s'_o$$

is not an integer. Since

$$d^{(k_1)}_{o,o} = e_o = h_o$$

is an integer, it follows from the rule of choosing an equation in the iterative process that the equation

$$x_1 = d^{(k_1)}_{1,0} + \sum_{j=1}^{q} d^{(k_1)}_{1,j} w^{(k_1)}_j$$

is to be chosen to construct the new equation

$$x_{p+q+k_1+1} = -s'_o + \sum_{j=1}^{q} s'_j w^{(k_1)}_j \tag{45}$$

where

$$s'_j = d^{(k_1)}_{1,j} - h'_j, \quad (j = 1, 2, \ldots, q)$$

with h'_j standing for the largest integer not greater than $d^{(k_1)}_{1,j}$.

Next we adjoin the equation (45) to the system (28) and (29) with $k = k_1$ and apply the dual-simplex method to the system (28), (29), and (45). By assumption, we can obtain an optimal solution given by the vector $\delta_o^{(k_1+1)}$. Since $-s_o'$ is the only negative constant term in the system (28), (29), and (45), the first pivoting in the dual-simplex method is to replace one of the nonbasic variables $w_1^{(k_1)}, \ldots, w_q^{(k_1)}$ by the new variable x_{p+q+k_1+1}. If, according to the rule of the dual-simplex method, the variable $w_{j_o}^{(k_1)}$ is selected to be replaced by x_{p+q+k_1+1}, then the equations (28), (29), and (45) will be transformed into the form (37) and (38) with $i = 1, 2, \ldots, p+q+k_1+1$, where the new nonbasic variables $z_1, \ldots, z_q$ are given by

$$z_j = \begin{cases} w_j^{(k_1)}, & (\text{if } j \neq j_o) \\ x_{p+q+k_1+1}, & (\text{if } j = j_o) \end{cases}$$

By the pivoting process, we have

$$f_{o,o} = d_{o,o}^{(k_1)} + \frac{s_o'}{s_{j_o}'} d_{o,j_o}^{(k_1)} \tag{46}$$

On the other hand, we have

$$d_{o,o}^{(k_1)} \leqq f_{o,o} \leqq d_{o,o}^{(k_1+1)} = d_{o,o}^{(k_1)} \tag{47}$$

Since $s_o' \neq 0$, (46) and (47) imply

$$d_{o,j_o}^{(k_1)} = 0 \tag{48}$$

Because of the dual feasibility, (48) implies

$$d_{1,j_o}^{(k_1)} \geqq 0 \tag{49}$$

This fact (49) implies

$$s'_{j_o} \leqq d_{1,j_o}^{(k_1)} \tag{50}$$

Then, by the pivoting process, we have

$$\begin{aligned} f_{1,0} &= d_{1,0}^{(k_1)} + \frac{s'_o}{s'_{j_o}} d_{1,j_o}^{(k_1)} \\ &\geqq d_{1,0}^{(k_1)} + s'_o = h'_o \end{aligned}$$

because of (50). On the other hand, since

$$f_{1,0} \leqq d_{1,0}^{(k)} \leqq e_1 \leqq h'_o, \quad (k > k_1)$$

it follows that

$$f_{1,0} = d_{1,0}^{(k)} = e_1 = h'_o, \quad (k > k_1)$$

Thus, we have proved that the limit e_1 of (43) is an integer and that there exists an integer $k_1 > k_0$ such that

$$d_{1,0}^{(k)} = e_1, \quad (k > k_1)$$

We can now repeat the same argument given above for $i = 2, 3, \ldots, p+q$ or apply the mathematical induction. Hence, for every $i = 0, 1, \ldots, p+q$, the sequence

$$d_{i,o}^{(o)}, \; d_{i,o}^{(1)}, \ldots, d_{i,o}^{(k)}, \ldots$$

converges to a limit

$$\lim_{k \to \infty} d_{i,o}^{(k)} = e_i$$

where e_i is an integer; furthermore, there exists an integer k_i such that

$$d_{i,o}^{(k)} = e_i, \quad (k > k_i)$$

Finally, let k be a finite integer such that

$$k > k_i, \quad (i = 0, 1, \ldots, p+q)$$

Then it follows that

$$y = d_{o,o}^{(k)} = e_o$$

$$x_i = d_{i,o}^{(k)} = e_i, \quad (i = 1, 2, \ldots, p+q)$$

is an integral optimal solution and the process terminates after k iterations. This completes the finiteness proof.

Section 5
AN ILLUSTRATIVE EXAMPLE

In this section, we shall study the standard integral minimum program of finding nonnegative integers

$$t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}$$

which minimize a given linear function

$$y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 + t_{10}$$

subject to a system of 20 linear inequalities:

$$t_{10} - t_3 - t_9 \geqq 0$$

$$t_{10} - t_5 - t_8 \geqq 0$$

$$t_{10} - t_2 - t_3 - t_8 \geqq 0$$

$$t_{10} - t_2 - t_4 - t_5 \geqq 0$$

$$t_{10} - t_1 - t_2 - t_3 - t_5 \geqq 0$$

$$t_4 + t_9 - t_{10} - 1 \geqq 0$$

$$t_6 + t_7 - t_{10} - 1 \geqq 0$$

$$t_1 + t_2 + t_9 - t_{10} - 1 \geqq 0$$

$$t_1 + t_4 + t_6 - t_{10} - 1 \geqq 0$$

$$t_3 + t_4 + t_5 - t_{10} - 1 \geqq 0$$

$$t_1 + t_2 + t_3 + t_6 - t_{10} - 1 \geqq 0$$

$$t_1 + t_2 + t_4 + t_5 - t_{10} - 1 \geqq 0$$

$$t_2 - t_1 \geq 0$$
$$t_3 - t_2 \geq 0$$
$$t_4 - t_3 \geq 0$$
$$t_5 - t_4 \geq 0$$
$$t_6 - t_5 \geq 0$$
$$t_7 - t_6 \geq 0$$
$$t_8 - t_7 \geq 0$$
$$t_9 - t_8 \geq 0$$

Next, we introduce slack variables x_i, $(1 \leq i \leq 20)$, for the left members of these inequalities. Thus, the given problem is reduced to that of finding nonnegative integers y, x_i, and t_j, which minimize y subject to the following twenty-one equations:

$$y = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 + t_{10}$$
$$x_1 = t_{10} - t_3 - t_9$$
$$x_2 = t_{10} - t_5 - t_8$$
$$x_3 = t_{10} - t_2 - t_3 - t_8$$
$$x_4 = t_{10} - t_2 - t_4 - t_5$$
$$x_5 = t_{10} - t_1 - t_2 - t_3 - t_5$$
$$x_6 = t_4 + t_9 - t_{10} - 1$$
$$x_7 = t_6 + t_7 - t_{10} - 1$$
$$x_8 = t_1 + t_2 + t_9 - t_{10} - 1$$
$$x_9 = t_1 + t_4 + t_6 - t_{10} - 1$$
$$x_{10} = t_3 + t_4 + t_5 - t_{10} - 1$$
$$x_{11} = t_1 + t_2 + t_3 + t_6 - t_{10} - 1$$
$$x_{12} = t_1 + t_2 + t_4 + t_5 - t_{10} - 1$$

$$x_{13} = t_2 - t_1$$
$$x_{14} = t_3 - t_2$$
$$x_{15} = t_4 - t_3$$
$$x_{16} = t_5 - t_4$$
$$x_{17} = t_6 - t_5$$
$$x_{18} = t_7 - t_6$$
$$x_{19} = t_8 - t_7$$
$$x_{20} = t_9 - t_8$$

Since all coefficients of the linear function y are positive, the problem is dually feasible; hence, we can apply the dual-simplex method and write

$$x_{20+j} = t_j, \qquad (1 \leqq j \leqq 10)$$

Then we can write the initial tableau T_o.

For definiteness, we choose the first variable with a negative constant term as the new nonbasic variable in each pivoting operation.

After twelve pivoting operations, we obtain a nonintegral optimal solution.
The thirteen tableaux of these operations are given as follows. The pivot element in each tableau is circled.

Tableau T_0

	c	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
y		1	1	1	1	1	1	1	1	1	1
x_1				-1						-1	1
x_2						-1			-1		1
x_3			-1	-1					-1		1
x_4			-1		-1	-1					1
x_5		-1	-1	-1		-1					1
x_6	-1				1					1	-1
x_7	-1						1	1			-1
x_8	-1	1	1							1	-1
x_9	-1	1			1		1				-1
x_{10}	-1			1	1	1					-1
x_{11}	-1	1	1	1			1				-1
x_{12}	-1	1	1		1	1					-1
x_{13}		-1	1								
x_{14}			-1	1							
x_{15}				-1	1						
x_{16}					-1	1					
x_{17}						-1	1				
x_{18}							-1	1			
x_{19}								-1	1		
x_{20}									-1	1	
x_{21}		1									
x_{22}			1								
x_{23}				1							
x_{24}					1						
x_{25}						1					
x_{26}							1				
x_{27}								1			
x_{28}									1		
x_{29}										1	
x_{30}											1

Tableau T_1

	c	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	x_6	t_{10}
y	1	1	1	1		1	1	1	1	1	2
x_1	-1			-1	1					-1	
x_2						-1			-1		1
x_3			-1	-1					-1		1
x_4			-1		-1	-1					1
x_5		-1	-1	-1		-1					1
x_6										1	
x_7	-1						1	1			-1
x_8		1	1		-1					1	
x_9	-1	1			1		1				-1
x_{10}	-1			1	1	1					-1
x_{11}	-1	1	1	1			1				-1
x_{12}	-1	1	1		1	1					-1
x_{13}		-1	1								
x_{14}			-1	1							
x_{15}				-1	1						
x_{16}					-1	1					
x_{17}						-1	1				
x_{18}							-1	1			
x_{19}								-1	1		
x_{20}	1				-1				-1	1	1
x_{21}		1									
x_{22}			1								
x_{23}				1							
x_{24}					1						
x_{25}						1					
x_{26}							1				
x_{27}								1			
x_{28}									1		
x_{29}	1				-1					1	1
x_{30}											1

Tableau T_2

	c	t_1	t_2	t_3	x_1	t_5	t_6	t_7	t_8	x_6	t_{10}
y	1	1	1	1		1	1	1	1	1	2
x_1					1						
x_2						-1			-1		1
x_3			-1	-1					-1		1
x_4	-1		-1	-1	-1	-1				-1	1
x_5		-1	-1	-1		-1					1
x_6										1	
x_7	-1						1	1			-1
x_8	-1	1	1	-1	-1						
x_9		1		1	1		1			1	-1
x_{10}				2	1	1				1	-1
x_{11}	-1	1	1	1			1				-1
x_{12}		1	1	1	1	1				1	-1
x_{13}		-1	1								
x_{14}			-1	1							
x_{15}	1				1					1	
x_{16}	-1			-1	-1	1				-1	
x_{17}						-1	1				
x_{18}							-1	1			
x_{19}								-1	1		
x_{20}				-1	-1				-1		1
x_{21}		1									
x_{22}			1								
x_{23}				1							
x_{24}	1			1	1					1	
x_{25}						1					
x_{26}							1				
x_{27}								1			
x_{28}									1		
x_{29}				-1	-1						1
x_{30}											1

Tableau T_3

	c	t_1	t_2	t_3	x_1	t_5	t_6	t_7	t_8	x_6	x_4
y	3	1	3	3	2	3	1	1	1	3	2
x_1					1						
x_2	1		1	1	1				-1	1	1
x_3	1				1	1			-1	1	1
x_4											1
x_5	1	-1			1					1	1
x_6										1	
x_7	-2		-1	-1	-1	-1	1	1		-1	-1
x_8	-1	1	1	-1	-1						
x_9	-1	1	-1			-1	1				-1
x_{10}	-1		-1	1							-1
x_{11}	-2	1			-1	-1	1			-1	-1
x_{12}	-1	1									-1
x_{13}		-1	1								
x_{14}			-1	1							
x_{15}	1				1					1	
x_{16}	-1			-1	-1	1				-1	
x_{17}						-1	1				
x_{18}							-1	1			
x_{19}								-1	1		
x_{20}	1		1			1			-1	1	1
x_{21}		1									
x_{22}			1								
x_{23}				1							
x_{24}	1			1	1					1	
x_{25}						1					
x_{26}							1				
x_{27}								1			
x_{28}									1		
x_{29}	1		1			1				1	1
x_{30}	1		1	1	1	1				1	1

Tableau T_4

	c	t_1	t_2	t_3	x_1	t_5	t_6	x_7	t_8	x_6	x_4
y	5	1	4	4	3	4		1	1	4	3
x_1					1						
x_2	1		1	1	1				-1	1	1
x_3	1				1	1			-1	1	1
x_4											1
x_5	1	-1			1					1	1
x_6										1	
x_7								1			
x_8	-1	1	1	-1	-1						
x_9	-1	1	-1			-1	1				-1
x_{10}	-1		-1	1							-1
x_{11}	-2	1			-1	-1	1			-1	-1
x_{12}	-1	1									-1
x_{13}		-1	1								
x_{14}			-1	1							
x_{15}	1				1					1	
x_{16}	-1			-1	-1	1				-1	
x_{17}						-1	1				
x_{18}	2		1	1	1	1	-2	1		1	1
x_{19}	-2		-1	-1	-1	-1	1	-1	1	-1	-1
x_{20}	1		1			1			-1	1	1
x_{21}		1									
x_{22}			1								
x_{23}				1							
x_{24}	1			1	1					1	
x_{25}						1					
x_{26}							1				
x_{27}	2		1	1	1	1	-1	1		1	1
x_{28}									1		
x_{29}	1		1			1				1	1
x_{30}	1		1	1	1	1				1	1

Tableau T_5

	c	x_8	t_2	t_3	x_1	t_5	t_6	x_7	t_8	x_6	x_4
y	6	1	3	5	4	4		1	1	4	3
x_1					1						
x_2	1		1	1	1				-1	1	1
x_3	1				1	1			-1	1	1
x_4											1
x_5		-1	1	-1						1	1
x_6										1	
x_7								1			
x_8		1									
x_9		1	-2	1	1	-1	1				-1
x_{10}	-1		-1	1							-1
x_{11}	-1	1	-1	1		-1	1			-1	-1
x_{12}		1	-1	1	1						-1
x_{13}	-1	-1	2	-1	-1						
x_{14}			-1	1							
x_{15}	1				1					1	
x_{16}	-1			-1	-1	1				-1	
x_{17}						-1	1				
x_{18}	2		1	1	1	1	-2	1		1	1
x_{19}	-2		-1	-1	-1	-1	1	-1	1	-1	-1
x_{20}	1		1			1			-1	1	1
x_{21}	1	1	-1	1	1						
x_{22}			1								
x_{23}				1							
x_{24}	1			1	1					1	
x_{25}						1					
x_{26}							1				
x_{27}	2		1	1	1	1	-1	1		1	1
x_{28}									1		
x_{29}	1		1			1				1	1
x_{30}	1		1	1	1	1				1	1

Tableau T_6

	c	x_8	t_2	x_{10}	x_1	t_5	t_6	x_7	t_8	x_6	x_4
y	11	1	8	5	4	4		1	1	4	8
x_1					1						
x_2	2		2	1	1				-1	1	2
x_3	1				1	1			-1	1	1
x_4											1
x_5	-1	-1		-1						1	
x_6										1	
x_7								1			
x_8		1									
x_9	1	1	-1	1	1	-1	1				
x_{10}				1							
x_{11}		1		1		-1	1			-1	
x_{12}	1	1		1	1						
x_{13}	-2	-1	1	-1	-1						-1
x_{14}	1			1							1
x_{15}	1				1					1	
x_{16}	-2		-1	-1	-1	1				-1	-1
x_{17}						-1	1				
x_{18}	3		2	1	1	1	-2	1		1	2
x_{19}	-3		-2	-1	-1	-1	1	-1	1	-1	-2
x_{20}	1		1			1			-1	1	1
x_{21}	2	1		1	1						1
x_{22}			1								
x_{23}	1		1	1							1
x_{24}	2		1	1	1					1	1
x_{25}						1					
x_{26}							1				
x_{27}	3		2	1	1	1	-1	1		1	2
x_{28}									1		
x_{29}	1		1			1				1	1
x_{30}	2		2	1	1	1				1	2

Tableau T_7

	c	x_8	t_2	x_{10}	x_1	t_5	t_6	x_7	t_8	x_5	x_4
y	15	5	8	9	4	4		1	1	4	8
x_1					1						
x_2	3	1	2	2	1				-1	1	2
x_3	2	1		1	1	1			-1	1	1
x_4											1
x_5										1	
x_6	1	1		1						1	
x_7								1			
x_8		1									
x_9	1	1	-1	1	1	-1	1				
x_{10}				1							
x_{11}	-1					-1	1			-1	
x_{12}	1	1		1	1						
x_{13}	-2	-1	1	-1	-1						-1
x_{14}	1			1							1
x_{15}	2	1		1	1					1	
x_{16}	-3	-1	-1	-2	-1	1				-1	-1
x_{17}						-1	1				
x_{18}	4	1	2	2	1	1	-2	1		1	2
x_{19}	-4	-1	-2	-2	-1	-1	1	-1	1	-1	-2
x_{20}	2	1	1	1		1			-1	1	1
x_{21}	2	1		1	1						1
x_{22}			1								
x_{23}	1		1	1							1
x_{24}	3	1	1	2	1					1	1
x_{25}						1					
x_{26}							1				
x_{27}	4	1	2	2	1	1	-1	1		1	2
x_{28}									1		
x_{29}	2	1	1	1		1				1	1
x_{30}	3	1	2	2	1	1				1	2

Tableau T_8

	c	x_8	t_2	x_{10}	x_1	t_5	x_{11}	x_7	t_8	x_5	x_4
y	15	5	8	9	4	4		1	1	4	8
x_1					1						
x_2	3	1	2	2	1				-1	1	2
x_3	2	1		1	1	1			-1	1	1
x_4											1
x_5										1	
x_6	1	1		1						1	
x_7								1			
x_8		1									
x_9	2	1	-1	1	1		1			1	
x_{10}				1							
x_{11}							1				
x_{12}	1	1		1	1						
x_{13}	-2	-1	1	-1	-1						-1
x_{14}	1			1							1
x_{15}	2	1		1	1					1	
x_{16}	-3	-1	-1	-2	-1	1				-1	-1
x_{17}	1						1			1	
x_{18}	2	1	2	2	1	-1	-2	1		-1	2
x_{19}	-3	-1	-2	-2	-1		1	-1	1		-2
x_{20}	2	1	1	1		1			-1	1	1
x_{21}	2	1		1	1						1
x_{22}			1								
x_{23}	1		1	1							1
x_{24}	3	1	1	2						1	1
x_{25}					1	1					
x_{26}	1					1	1			1	
x_{27}	3	1	2	2	1		-1	1			2
x_{28}									1		
x_{29}	2	1	1	1		1				1	1
x_{30}	3	1	2	2	1	1				1	2

Tableau T_9

	c	x_8	x_{13}	x_{10}	x_1	t_5	x_{11}	x_7	t_8	x_5	x_4
y	31	13	8	17	12	4		1	1	4	16
x_1					1						
x_2	7	3	2	4	3				-1	1	4
x_3	2	1		1	1	1			-1	1	1
x_4											1
x_5										1	
x_6	1	1		1						1	
x_7								1			
x_8		1									
x_9			-1				1			1	-1
x_{10}				1							
x_{11}							1				
x_{12}	1	1		1	1						
x_{13}			1								
x_{14}	1			1							1
x_{15}	2	1		1	1					1	
x_{16}	-5	-2	-1	-3	-2	1				-1	-2
x_{17}	1						1			1	
x_{18}	6	3	2	4	3	-1	-2	1		-1	4
x_{19}	-7	-3	-2	-4	-3		1	-1	1		-4
x_{20}	4	2	1	2	1	1			-1	1	2
x_{21}	2	1		1	1						1
x_{22}	2	1	1	1	1						1
x_{23}	3	1	1	2	1						2
x_{24}	5	2	1	3	1					1	2
x_{25}					1	1					
x_{26}	1					1	1			1	
x_{27}	7	3	2	4	3		-1	1			4
x_{28}									1		
x_{29}	4	2	1	2	1	1				1	2
x_{30}	7	3	2	4	3	1				1	4

Tableau T_{10}

	c	x_8	x_{13}	x_{10}	x_1	x_{16}	x_{11}	x_7	t_8	x_5	x_4
y	51	21	12	29	20	4		1	1	8	24
x_1					1						
x_2	7	3	2	4	3				-1	1	4
x_3	7	3	1	4	3	1			-1	2	3
x_4											1
x_5										1	
x_6	1	1		1				1		1	
x_7											
x_8		1									
x_9			-1				1			1	-1
x_{10}				1							
x_{11}							1				
x_{12}	1	1		1	1						
x_{13}			1								
x_{14}	1			1	1						1
x_{15}	2	1		1						1	
x_{16}						1					
x_{17}	1						1			1	
x_{18}	1	1	1	1	1	-1	-2	1		-2	2
x_{19}	-7	-3	-2	-4	-3		1	-1	1		-4
x_{20}	9	4	2	5	3	1			-1	2	4
x_{21}	2	1		1	1						1
x_{22}	2	1	1	1	1						1
x_{23}	3	1	1	2	1						2
x_{24}	5	2	1	3	1					1	2
x_{25}	5	2	1	3	3	1				1	2
x_{26}	6	2	1	3	2	1	1			2	2
x_{27}	7	3	2	4	3		-1	1			4
x_{28}									1		
x_{29}	9	4	2	5	3	1				2	4
x_{30}	12	5	3	7	5	1				2	6

Tableau T_{11}

	c	x_8	x_{13}	x_{10}	x_1	x_{16}	x_{19}	x_7	t_8	x_5	x_4
y	51	21	12	29	20	4		1	1	8	24
x_1					1						
x_2	7	3	2	4	3				-1	1	4
x_3	7	3	1	4	3	1			-1	2	3
x_4											1
x_5										1	
x_6	1	1	1	1				1		1	
x_7											
x_8		1									
x_9	7	3	1	4	3		1	1	-1	1	3
x_{10}				1							
x_{11}	7	3	2	4	3		1	1	-1		4
x_{12}	1	1		1	1						
x_{13}			1								
x_{14}	1			1	1						1
x_{15}	2	1		1						1	
x_{16}						1					
x_{17}	8	3	2	4	3		1	1	-1	1	4
x_{18}	-13	-5	-3	-7	-5	-1	-2	-1	2	-2	-6
x_{19}							1				
x_{20}	9	4	2	5	3	1			-1	2	4
x_{21}	2	1		1	1						1
x_{22}	2	1	1	1	1						1
x_{23}	3	1	1	2	1						2
x_{24}	5	2	1	3	1					1	2
x_{25}	5	2	1	3	3	1				1	2
x_{26}	13	5	3	7	5	1	1	1	-1	2	6
x_{27}							-1		1		
x_{28}									1		
x_{29}	9	4	2	5	3	1				2	4
x_{30}	12	5	3	7	5	1				2	6

Tableau T_{12}

	c	x_8	x_{13}	x_{10}	x_1	x_{16}	x_{19}	x_7	x_{18}	x_5	x_4
y	57.5	23.5	13.5	32.5	22.5	4.5	1	1.5	.5	9	27
x_1					1						
x_2	.5	.5	.5	.5	.5	.5	-1	-.5	-.5		1
x_3	.5	.5	-.5	.5	.5	.5	-1	-.5	-.5	1	
x_4											1
x_5										1	
x_6	1	1	1	1				1		1	
x_7											
x_8		1									
x_9	.5	.5	-.5	.5	.5	-.5		.5	-.5		
x_{10}				1							
x_{11}	.5	.5	.5	.5	.5	-.5		.5	-.5	-1	1
x_{12}	1	1		1	1						
x_{13}			1								
x_{14}	1			1	1						1
x_{15}	2	1		1						1	
x_{16}						1					
x_{17}	1.5	.5	.5	.5	.5	-.5		.5	-.5		1
x_{18}									1		
x_{19}							1				
x_{20}	2.5	1.5	.5	1.5	.5	.5	-1	-.5	-.5	1	1
x_{21}	2	1		1	1						1
x_{22}	2	1	1	1	1						1
x_{23}	3	1	1	2	1						2
x_{24}	5	2	1	3	1					1	2
x_{25}	5	2	1	3	3	1				1	2
x_{26}	6.5	2.5	1.5	3.5	2.5	.5		.5	-.5	1	3
x_{27}	6.5	2.5	1.5	3.5	2.5	.5		.5	.5	1	3
x_{28}	6.5	2.5	1.5	3.5	2.5	.5	1	.5	.5	1	3
x_{29}	9	4	2	5	3	1				2	4
x_{30}	12	5	3	7	5	1				2	6

Since the constant column in T_{12} is nonnegative, we obtain an optimal solution

$$
\begin{aligned}
t_1 &= x_{21} = 2\\
t_2 &= x_{22} = 2\\
t_3 &= x_{23} = 3\\
t_4 &= x_{24} = 5\\
t_5 &= x_{25} = 5\\
t_6 &= x_{26} = 6.5\\
t_7 &= x_{27} = 6.5\\
t_8 &= x_{28} = 6.5\\
t_9 &= x_{29} = 9\\
t_{10} &= x_{30} = 12
\end{aligned}
$$

The minimum of the linear function

$$y = \sum_{j=1}^{10} t_j$$

for all feasible solutions is 57.5. This optimal solution is not integral; in fact, it is obvious that no integral feasible solution can give y the nonintegral value 57.5.

To find an integral optimal solution for our problem, let us apply the process described in section 3.

Since the leading element 57.5 in the constant column is nonintegral, we choose the equation

$$\begin{aligned} y = 57.5 &+ 23.5\, x_8 + 13.5\, x_{13} + 32.5\, x_{10} + 22.5\, x_1 \\ &+ 1.5\, x_{16} + x_{19} + 1.5\, x_7 + .5\, x_{18} + 9\, x_5 + 27\, x_4 \end{aligned}$$

in the construction of the new equation. Thus, we obtain the new equation

$$\begin{aligned} x_{31} = -.5 &+ .5\, x_8 + .5\, x_{13} + .5\, x_{10} \\ &+ .5\, x_1 + .5\, x_{16} + .5\, x_7 + .5\, x_{18} \end{aligned}$$

Adjoin a new row corresponding to this additional constraint to the last tableau T_{12}. Hence we obtain the following expanded tableau T_{13}.

In T_{13}, since $-.5$ is the only negative entry in the constant column, we have to use x_{31} as the new nonbasic variable. The pivot element is marked by a circle. After the pivoting operation, we obtain the tableau T_{14}.

Tableau T_{13}

	c	x_8	x_{13}	x_{10}	x_1	x_{16}	x_{19}	x_7	x_{18}	x_5	x_4
y	57.5	23.5	13.5	32.5	22.5	4.5	1	1.5	.5	9	27
x_1					1						
x_2	.5	.5	.5	.5	.5	-.5	-1	-.5	-.5		1
x_3	.5	.5	-.5	.5	.5	.5	-1	-.5	-.5	1	
x_4											1
x_5										1	
x_6	1	1	1	1				1		1	
x_7											
x_8		1									
x_9	.5	.5	-.5	.5	.5	-.5		.5	-.5		
x_{10}				1							
x_{11}	.5	.5	.5	.5	.5	-.5		.5	-.5	-1	1
x_{12}	1	1		1	1						
x_{13}			1								
x_{14}	1			1	1						1
x_{15}	2	1		1						1	
x_{16}						1					
x_{17}	1.5	.5	.5	.5	.5	-.5		.5	-.5		1
x_{18}									1		
x_{19}							1				
x_{20}	2.5	1.5	.5	1.5	.5	.5	-1	-.5	-.5	1	1
x_{21}	2	1		1	1						1
x_{22}	2	1	1	1	1						1
x_{23}	3	1	1	2	1						2
x_{24}	5	2	1	3	1					1	2
x_{25}	5	2	1	3	3	1				1	2
x_{26}	6.5	2.5	1.5	3.5	2.5	.5		.5	-.5	1	3
x_{27}	6.5	2.5	1.5	3.5	2.5	.5		.5	.5	1	3
x_{28}	6.5	2.5	1.5	3.5	2.5	.5	1	.5	.5	1	3
x_{29}	9	4	2	5	3	1				2	4
x_{30}	12	5	3	7	5	1				2	6
x_{31}	-.5	.5	.5	.5	.5	.5		.5	.5		

Tableau T_{14}

	c	x_8	x_{13}	x_{10}	x_1	x_{16}	x_{19}	x_7	x_{31}	x_5	x_4
y	58	23	13	32	22	4	1	1	1	9	27
x_1					1						
x_2		1	1	1	1		-1		-1		1
x_3		1		1	1	1	-1		-1	1	
x_4											1
x_5										1	
x_6	1	1	1	1				1		1	
x_7											
x_8		1									
x_9		1		1	1			1	-1		
x_{10}				1							
x_{11}		1	1	1	1			1	-1	-1	1
x_{12}	1	1		1	1						
x_{13}			1								
x_{14}	1			1	1						1
x_{15}	2	1		1						1	
x_{16}						1					
x_{17}	1	1	1	1	1			1	-1		1
x_{18}	1	-1	-1	-1	-1	-1		-1	2		
x_{19}							1				
x_{20}	2	2	1	2	1	1	-1		-1	1	1
x_{21}	2	1		1	1						1
x_{22}	2	1	1	1	1						1
x_{23}	3	1	1	2	1						2
x_{24}	5	2	1	3	1					1	2
x_{25}	5	2	1	3	3	1				1	2
x_{26}	6	3	2	4	3	1		1	-1	1	3
x_{27}	7	2	1	3	2				1	1	3
x_{28}	7	2	1	3	2		1		1	1	3
x_{29}	9	4	2	5	3	1				2	4
x_{30}	12	5	3	7	5	1				2	6
x_{31}									1		

Now, in the tableau T_{14}, all entries are nonnegative integers. Hence, we obtain an optimal integral solution

$$
\begin{aligned}
t_1 &= x_{21} = 2 \\
t_2 &= x_{22} = 2 \\
t_3 &= x_{23} = 3 \\
t_4 &= x_{24} = 5 \\
t_5 &= x_{25} = 5 \\
t_6 &= x_{26} = 6 \\
t_7 &= x_{27} = 7 \\
t_8 &= x_{28} = 7 \\
t_9 &= x_{29} = 9 \\
t_{10} &= x_{30} = 12
\end{aligned}
$$

The minimum of the linear function

$$y = \sum_{j=1}^{10} t_i$$

for all integral feasible solutions is 58.

Section 6
SWITCHING FUNCTIONS

Let Q denote the set that consists of the two integers 0 and 1. For any given integer $n \geqq 1$, consider the Cartesian power

$$Q^n = Q \times \ldots \times Q$$

which is the Cartesian product of n copies of Q. Thus, the elements of Q^n are the 2^n ordered n-tuples

$$(x_1, x_2, \ldots, x_n)$$

where the k^{th} coordinate x_k is in Q for every $k = 1, 2, \ldots, n$. Hereafter, Q^n will be called the n-<u>cube</u> and its 2^n elements will be called its <u>points.</u>

By a <u>switching function</u> of n variables, we mean any subset F of the n-cube Q^n. Since Q^n has 2^n points, there are 22^n different switching functions of n variables.

A switching function F of n variables is said to be <u>linearly separable</u> provided that there exist $n + 1$ real numbers $w_1, w_2, \ldots, w_n, w_{n+1}$ such that, for every point

$$x = (x_1, \ldots, x_n) \in Q^n$$

we have $x \in F$ if and only if

$$w_1 x_1 + \ldots + w_n x_n \leqq w_{n+1}$$

The set $W = (w_1, \ldots, w_n, w_{n+1})$ is called a <u>separating system</u> of F; the real numbers $w_1, \ldots, w_n$ are called the <u>weights</u>, and the real number w_{n+1} is called

the <u>threshold.</u> By taking the threshold w_{n+1} as small as possible while the weights $w_1, \ldots, w_n$ are held fixed, we may assume that, in case F is not empty, there exists a point $x = (x_1, \ldots, x_n)$ in F such that

$$w_1 x_1 + \ldots + w_n x_n = w_{n+1}$$

Consider the complement

$$F' = Q^n - F$$

of F. For every point $y = (y_1, \ldots, y_n)$ in F', we have

$$w_1 y_1 + \ldots + w_n y_n > w_{n+1}$$

Let M denote the minimal value of

$$w_1 y_1 + \ldots + w_n y_n - w_{n+1}$$

for all points $y = (y_1, \ldots, y_n)$ in F' in case F' is not empty. This positive real number M is called the <u>margin</u> of the separating system W (Ref. 15, p. 6). A separating system $W = (w_1, \ldots, w_n, w_{n+1})$ of F is said to be <u>normal</u> provided that M = 1. Every separating system

$$W = (w_1, \ldots, w_n, w_{n+1})$$

of F can be <u>normalized</u> by dividing each w_i, $(i = 1, \ldots, n+1)$, by the margin M of W. Precisely, the set

$$W' = (w_1', \ldots, w_n', w_{n+1}')$$

with $w_i' = w_i / M$ for each $i = 1, \ldots, n+1$ is a normal separating system of F. Hence, every linearly separable switching function F has a normal separating system.

Let $W = (w_1, \ldots, w_n, w_{n+1})$ be any normal separating system of a given linearly separable switching function F of n variables. Then, for every point

$$x = (x_1, \ldots, x_n) \epsilon Q^n$$

we have

$$w_1 x_1 + \cdots + w_n x_n \leqq w_{n+1}, \qquad (\text{if } x \epsilon F)$$

$$w_1 x_1 + \cdots + w_n x_n \geqq w_{n+1} + 1, \quad (\text{if } x \epsilon F')$$

By a canonical switching function of n variable, we mean a linearly separable switching function F of n variables which admits a separating system $W = (w_1, \ldots, w_n, w_{n+1})$ satisfying

$$0 \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_i \leqq w_{i+1} \leqq \cdots \leqq w_n$$

in words, the weights $w_1, \ldots, w_n$ in W are nonnegative and nondecreasing.

It is well known (Refs. 16 and 17) that every linearly separable switching function F of n variables can be reduced to a unique canonical switching function by permuting and complementing a number of the variables.

Section 7
REGULAR SWITCHING FUNCTIONS

By a <u>weight function</u> of n variables, we mean a homogeneous linear function

$$w : R^n \longrightarrow R$$

on the n-dimensional Euclidean space R^n with real values. Precisely, there are n real numbers $w_1, \ldots, w_n$ such that, for an arbitrary point $x = (x_1, \ldots, x_n)$ of R^n, we have

$$w(x) = w_1 x_1 + \ldots + w_n x_n$$

The real numbers $w_1, \ldots, w_n$ are called the <u>coefficients</u> of the weight function w.

A weight function $w : R^n \longrightarrow R$ with coefficients $w_1, \ldots, w_n$ is said to be <u>canonical</u> provided

$$0 \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_i \leqq w_{i+1} \leqq \cdots \leqq w_n$$

By means of the canonical weight functions of n variables, we can define a partial order in the n-cube Q^n as follows: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be any two points of Q^n; then we define

$$x \leqq y$$

if and only if

$$w(x) \leqq w(y)$$

for every canonical weight function w of n variables. This partial order in Q^n is called the canonical partial order (Ref. 17).

The following criterion for $\leqq$ was established in Ref. 17. For arbitrary points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of Q^n, we have $x \leqq y$ if and only if

$$\sum_{j=i}^{n} x_j \leqq \sum_{j=i}^{n} y_j$$

for every $i = 1, 2, \ldots, n$.

With respect to the canonical partial order in the n-cube Q^n, we define the maximal points and the minimal points of an arbitrarily given switching function F of n variables as follows. A point x of F is said to be maximal provided that, for an arbitrary point y of F, $x \leqq y$ implies $x = y$. Similarly, a point x of F is said to be minimal provided that, for an arbitrary point y of F, $y \leqq x$ implies $y = x$.

Using the canonical partial order $\leqq$ in the n-cube Q^n, we can define the regular switching functions. A switching function F of n variables is said to be regular if and only if it satisfies the following regularity condition:

If $x \in F$ and $y \leqq x$, then $y \in F$.

Obviously, every canonical switching of n variables is regular. In Ref. 18, it was proved that every regular switching function of $n \leqq 5$ variables is canonical; in Ref. 6, an example is given which shows that not every regular switching function of 6 variables is canonical.

Section 8
SYNTHESIS AND MINIMIZATION

Let F be an arbitrary switching function of n variables. The synthesis problem for the linear separability of F is to determine whether or not F is linearly separable and to find a separating system $(w_1, \ldots, w_n, w_{n+1})$ for F in case F is linearly separable.

Among various synthesis methods for linear separability introduced in the literature, the one given by D. G. Willis (Ref. 1) turns out to be the most convenient because it involves as few linear inequalities as possible. In Ref. 1, the synthesis problem for the linear separability of arbitrary switching functions of n variables was reduced to that of the regular switching functions of n variables. Thus, it remains to determine whether or not a given regular switching function F is linearly separable and to find a normal canonical separating system $(w_1, \ldots, w_n, w_{n+1})$ for F in case F is linearly separable and hence canonical.

For the convenience of the reader, we shall briefly describe the Willis synthesis method.

Let F be an arbitrary regular switching function of n variables. We assume that F is nontrivial, i.e.,

$$F \neq \square, \qquad F \neq Q^n$$

where $\square$ denotes the empty set. Let L denote the set of all maximal points of F; and let M denote the set of all minimal points of the complement $F' = Q^n - F$. Let

$$a_i = (a_{i1}, \ldots, a_{in}), \qquad (i = 1, 2, \ldots, p)$$

be the points of L and

$$b_j = (b_{j1}, \ldots, b_{jn}), \qquad (j = 1, 2, \ldots, q)$$

be the points of M. Consider the following system of $p + q + n$ linear inequalities:

$$\left.\begin{array}{l} \sum_{k=1}^{n} a_{ik} w_k \leqq w_{n+1}, \quad (i = 1, 2, \ldots, p) \\ \\ \sum_{k=1}^{n} b_{jk} w_k \geqq w_{n+1} + 1, \quad (j = 1, 2, \ldots, q) \end{array}\right\} \tag{51}$$

$$0 \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_n$$

Then, the Willis synthesis theorem states that the given regular switching function F is linearly separable if and only if the system (51) of linear inequalities has a solution (and hence an integral solution).

In our previous reports (Refs. 2–5), various methods were applied to solve the system (51).

The next problem is naturally the minimization problem, which is to find the most economical solution of the system (51) in case the given regular switching function F is linearly separable. In other words, the minimization problem is to find a solution

$$(w_1, \ldots, w_n, w_{n+1})$$

of the system (51) which makes some cost function $\phi(w_1, \ldots, w_n, w_{n+1})$ minimal.

For the minimization problem, let us first pick up the cost function ϕ . Assume that the cost of realizing the w_i is proportional to the magnitude of w_i for each $i = 1, 2, \ldots, n+1$. Under this assumption, the cost function ϕ will be a homogeneous linear function.

$$\phi\ (w_1, \ldots, w_n, w_{n+1}) = \sum_{i=1}^{n+1} \gamma_i w_i , \tag{52}$$

where the coefficients $\gamma_1, \ldots, \gamma_{n+1}$ are nonnegative real numbers. In the literature (see Ref. 8), two different cost functions have been studied; one of these is defined by $\gamma_i = 1$ for all $i = 1, 2, \ldots, n, n+1$ and the other is given by $\gamma_i = 1$ for $i = 1, 2, \ldots, n$ and $\gamma_{n+1} = 0$.

Having fixed the cost function ϕ by (52), the minimization problem for a given regular switching function F is that of finding a normal canonical separating system that minimizes ϕ ; in other words, the minimization problem is the standard linear program of finding nonnegative real numbers.

$$w_1, w_2, \ldots, w_n, w_{n+1}$$

which minimizes the cost function (52) and satisfy the system (51), or equivalently the following system of $m = p + q + n - 1$ linear inequalities:

$$\left.\begin{array}{ll}
- \sum_{k=1}^{n} a_{ik} w_k + w_{n+1} \geqq 0, & (i = 1, 2, \ldots, p) \\
\\
- 1 + \sum_{k=1}^{n} b_{jk} w_k - w_{n+1} \geqq 0, & (j = 1, 2, \ldots, q) \\
\\
w_{k+1} - w_k \geqq 0, & (k = 1, 2, \ldots, n+1)
\end{array}\right\} \tag{53}$$

In an earlier report (Ref. 19), it was proved under the assumption

$$\gamma_n > 0$$

that this minimization problem for F has optimal solutions in case it is feasible. In other words, if $\gamma_n > 0$, then every linearly separable regular switching function F of n variables has a minimal normal canonical separating system.

Various algorithms for computing these minimal normal canonical separating systems were proposed in our recent reports (Refs. 6 and 7). In particular, if

$$\gamma_i > 0 , \quad (i = 1, 2, \ldots, n) \tag{54}$$

the minimization problem is dually feasible for every nonempty regular switching function F and hence the dual-simplex method can be very efficiently applied without the work of finding a feasible solution first (see Ref. 7).

Section 9
INTEGRAL MINIMIZATION PROBLEM

Throughout the present section, let us assume that the cost function (52) satisfies the condition (54) and that the given regular switching function F of n variables is nontrivial, i.e.,

$$F \neq \square \ , \quad F \neq Q^n$$

For most of the threshold devices realizing linearly separable switching functions, the numbers

$$w_1, w_2, \ldots, w_n, w_{n+1}$$

are required to be integers. Hence, we want to find nonnegative integers w_i, $(i = 1, 2, \ldots, n+1)$, which minimize the cost function (52) and satisfy the system (53) of linear inequalities. In this integral minimization problem, the coefficients $\gamma_1, \ldots, \gamma_{n+1}$ are assumed to be nonnegative integers. Hence, the value of the cost function (52) for any integral feasible solution of the problem is also a nonnegative integer.

The optimal solutions of the minimization problem in the preceding section may be not integral. For example, let us consider the regular switching function

$$F = 987643/651/51//2$$

of nine variables in the notation introduced in our earlier report (Ref. 20). This regular switching function F consists of 58 points of the 9-cube Q^9, and its complement F' consists of $2^9 - 58 = 454$ points of Q^9.

By the methods developed in Refs. 20 and 21, one can find the set L of all maximal points of F and the set M of all minimal points of its complement F'. Indeed, L consists of the following five points of Q^9

(0, 0, 1, 0, 0, 0, 0, 0, 1)
(0, 0, 0, 0, 1, 0, 0, 1, 0)
(0, 1, 1, 0, 0, 0, 0, 1, 0)
(0, 1, 0, 1, 1, 0, 0, 0, 0)
(1, 1, 1, 0, 1, 0, 0, 0, 0)

and M consists of the following seven points of Q^9

(0, 0, 0, 1, 0, 0, 0, 0, 1)
(0, 0, 0, 0, 0, 1, 1, 0, 0)
(1, 1, 0, 0, 0, 0, 0, 0, 1)
(1, 0, 0, 1, 0, 1, 0, 0, 0)
(0, 0, 1, 1, 1, 0, 0, 0, 0)
(1, 1, 1, 0, 0, 1, 0, 0, 0)
(1, 1, 0, 1, 1, 0, 0, 0, 0)

Assume the cost function to be the following linear function

$$y = w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10}$$

Then, one can easily see that our minimization problem reduces to the illustrative example in section 5 with $t_j = w_j$ for each $j = 1, 2, \ldots, 10$. Hence, after twelve pivoting operations, the dual-simplex method gives an optimal solution

$$\left.\begin{aligned} w_1 &= 2 \\ w_2 &= 2 \\ w_3 &= 3 \\ w_4 &= 5 \\ w_5 &= 5 \\ w_6 &= 6.5 \\ w_7 &= 6.5 \\ w_8 &= 6.5 \\ w_9 &= 9 \\ w_{10} &= 12 \end{aligned}\right\} \quad (55)$$

with the minimal total cost

$$y = 57.5$$

Hence, this regular switching function F is the one studied by D. G. Willis (Ref. 8).

Since the minimal total cost is 57.5 , no integral feasible solution can be an optimal solution. In fact, D. G. Willis proved that

$$(2,\ 2,\ 3,\ 5,\ 5,\ 6.5,\ 6.5,\ 6.5,\ 9,\ 12)$$

is the only minimal normal separating system for F.

Now, let us return to the general case described at the beginning of this section.

Apply the dual-simplex method to our minimization problem. The result will be that either F is not linearly separable or an optimal solution can be obtained from the final simplex tableau.

If the optimal solution obtained by the dual-simplex method consists of integers only, then we have obtained an integral normal canonical separating system which is minimal for all normal canonical separating systems of F. Thus, the problem of finding minimal integral weights and threshold is solved in this case.

Otherwise, if the optimal solution is not integral, then we can apply Gomery's method as described in section 3. It remains for us to verify that the finiteness proof in section 4 holds for this special case.

In case $\gamma_{n+1} > 0$, the condition (B) at the beginning of section 4 is satisfied since F is linearly separable and

$$\gamma_i > 0, \qquad (i = 1, 2, \ldots, n+1).$$

Hence, the finiteness proof holds for this case.

On the other hand, let $\gamma_{n+1} = 0$. Since $F \neq Q^n$, it follows that the unit point $(1, 1, \ldots, 1)$ of Q^n is in the complement F' and hence

$$w_{n+1} \leqq \sum_{i=1}^{n} w_i - 1$$

for every feasible solution $(w_1, \ldots, w_{n+1})$ of (53). Hence, in the finiteness proof, we can also prove that the sequence (41) is bounded. Therefore, the finiteness proof holds also for this case.

Thus, after a finite number of iterations of the process described in section 3, we will get a minimal integral solution of our minimization problem.

Let us consider the regular switching function

$$F = 987643/651/51//2$$

of nine variables again. We have observed that the dual-simplex method gives us the optimal solution (55) with cost $y = 57.5$. Since this solution is not integral, we apply Gomery's process to the resulting system. As shown in section 5, we obtain finally an optimal integral solution

$$\begin{aligned} w_1 &= 2 \\ w_2 &= 2 \\ w_3 &= 3 \\ w_4 &= 5 \\ w_5 &= 5 \\ w_6 &= 6 \\ w_7 &= 7 \\ w_8 &= 7 \\ w_9 &= 9 \\ w_{10} &= 12 \end{aligned}$$

with total cost

$$y = 58$$

Thus we have obtained a minimal integral normal canonical separating system

$$(2, 2, 3, 5, 5, 6, 7, 7, 9, 12)$$

of the given regular switching function F of nine variables; where

$$(2, 2, 3, 5, 5, 6, 7, 7, 9)$$

are the weights and 12 is the threshold.

Section 10
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